

# Pion and Kaon Decay Constants: Lattice vs. Resonance Chiral Theory

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## Abstract

The Lattice results for the pion and kaon decay constants are analysed within the Resonance Chiral Theory framework in the large  $N_C$  limit. The approximately linear behaviour of the observable at large light-quark mass is explained through the interaction with the lightest multiplet of scalar resonances. The analysis of the Lattice results allows to obtain the resonance mass  $M_S = 1049 \pm 25$  MeV and the Chiral Perturbation Theory parameters at leading order in  $1/N_C$ .

# 1 Introduction

Quantum Chromodynamics (QCD) has been proved to be the proper theory to describe the strong interactions. However in the low energy region the theory in terms of quarks and gluons becomes highly non perturbative. These degrees of freedom get confined within complex hadronic states. Below the first resonance multiplet ( $E \ll M_\rho$ ), the only particles in the spectrum are the light octet of pseudo-scalars, the pseudo-Nambu-Goldstone bosons (pNGB) from the spontaneous chiral symmetry breaking. At low momenta and small pNGB masses one may describe their interactions through a chiral invariant effective field theory, Chiral Perturbation Theory ( $\chi$ PT) [1]. It establishes an expansion on powers of the external momenta and masses over a characteristic chiral scale  $\Lambda_\chi \sim 4\pi F \sim 1.2$  GeV, being  $F \simeq F_\pi = 92.4$  MeV the physical pion decay constant.

The chiral expansion breaks down when either the momenta or the pNGB masses become large; when they approach to the  $\rho(770)$  mass the chiral expansion fails. The mesonic resonances can then be produced and their effects cannot be neglected any longer.

Alternatively, it is possible to employ the  $1/N_C$  expansion to describe the matrix elements –being  $N_C$  the number of colours– [2]. Resonance Chiral Theory (R $\chi$ T) [3] incorporates the interactions between the resonances and the Goldstones at leading order in  $1/N_C$  and also implements the chiral symmetry of the interaction. Likewise,  $\chi$ PT is fully recovered when going to low energies.

At leading order in  $1/N_C$  (LO) the observables are given by the tree-level amplitudes, being the mesonic loops suppressed by  $1/N_C$  [2]. In that situation R $\chi$ T is able to reproduce the short distance behaviour required by QCD for the pion form factors, two-point Green functions and forward scattering amplitudes [4, 5]. The  $1/N_C$  counting can also be carried to the next order (NLO) in a systematic way and the quantum loop corrections might be calculated [6].

The present study is focused on how the variation of the quark masses affects the pion and kaon decay constants,  $F_\pi$  and  $F_K$ , under the R $\chi$ T framework and the  $1/N_C$  expansion. Lattice calculations have provided information about QCD results for unphysical values of the  $u/d$  light quarks [7]. The simulations are forced sometimes to work with masses of nearly the size of the physical strange quark mass or higher. Thus, the usual  $\chi$ PT extrapolations break down and generate large unphysical chiral logarithms, yielding an large bending in the extrapolation [7, 8, 9]. Nonetheless, more than purely numerical values, this work aims providing a possible way to analyse the Lattice simulations at large quark masses, explaining why the usual linear extrapolations work so well and what are the underlying physical foundations.

Through the inclusion of the first resonance multiplets, with masses  $M_R \sim 1$  GeV, one expects to reproduce the physics for the Goldstones up to that range of momenta and masses.

Moreover, at LO in  $1/N_C$  only the scalar resonances contribute to  $F_\pi$  and  $F_K$ . Since the first multiplet of pseudo-scalar resonances is at much higher masses, it will not be considered in the calculation and its mixing with the pNGB will be neglected.

## 2 $R_\chi$ T lagrangian at LO in $1/N_C$

The leading order resonance lagrangian was developed in Ref. [3] and includes one multiplets of vector, axial-vector, scalar and pseudo-scalar resonances. It is composed by a pair of terms including only Goldstones –the  $\mathcal{O}(p^2)$   $\chi$ PT Lagrangian [1]–,

$$\mathcal{L}_2 = \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle, \quad (1)$$

and pieces including as well resonance fields. The brackets  $\langle \dots \rangle$  mean trace of the flavour matrices. The chiral tensors including the Goldstone fields can be obtained in Refs. [3, 4, 5] and they contain the external fields  $v^\mu$ ,  $a^\mu$ ,  $s$  and  $p$ , and the non-linear realization of the chiral symmetry given by the tensor  $u = \exp(i\Phi/\sqrt{2}F)$ , with the pNGB fields  $\Phi = \sum_a \phi_a \lambda_a / \sqrt{2}$ .

In the large  $N_C$  limit the  $q\bar{q}$  resonances form  $U(3)$  multiplets. The fields of a multiplet can be put together in a  $3 \times 3$  matrix which transforms linearly under the chiral symmetry,

$$R = \sum_a R_a \frac{\lambda_a}{\sqrt{2}}, \quad (2)$$

containing one chiral singlet field  $R_0$  and the remaining octet fields  $R_a$ , with  $a = 1, \dots, 8$ . All the fields  $R_a$  of the multiplet would be degenerate in the large  $N_C$  and massless quark limit. The kinetic terms are then constructed with these matrices:

$$\begin{aligned} \mathcal{L}_R^{\text{Kin}}(R = V, A) &= -\frac{1}{2} \langle \nabla^\lambda R_{\lambda\mu} \nabla_\nu R^{\nu\mu} - \frac{1}{2} M_R^2 R_{\mu\nu} R^{\mu\nu} \rangle, \\ \mathcal{L}_R^{\text{Kin}}(R = S, P) &= \frac{1}{2} \langle \nabla^\mu R \nabla_\mu R - M_R^2 R^2 \rangle, \end{aligned} \quad (3)$$

and the interaction lagrangian, linear in the resonance fields [3],

$$\begin{aligned} \mathcal{L}_{2V} &= \frac{F_V}{2\sqrt{2}} \langle V_{\mu\nu} f_+^{\mu\nu} \rangle + \frac{iG_V}{2\sqrt{2}} \langle V_{\mu\nu} [u^\mu, u^\nu] \rangle, \\ \mathcal{L}_{2A} &= \frac{F_A}{2\sqrt{2}} \langle A_{\mu\nu} f_-^{\mu\nu} \rangle, \\ \mathcal{L}_{2S} &= c_d \langle S u_\mu u^\mu \rangle + c_m \langle S \chi_+ \rangle, \\ \mathcal{L}_{2P} &= i d_m \langle P \chi_- \rangle, \end{aligned} \quad (4)$$

where the vector and axial-vector resonances are described by antisymmetric tensors.

### 3 Scalar tadpole and field redefinition

Analysing the LO lagrangian one observes the presence of a term linear in the scalar fields, i.e. a scalar tadpole. It is given by the term of  $\mathcal{L}_{2S}$  in Eq. (4) with the coupling  $c_m$ , which also provides the vertex for the scalar resonance production from a scalar quark current.

Chiral symmetry requires the quark masses to enter in the effective lagrangian only through the tensor  $\chi = 2B_0\{s(x) + ip(x)\}$ , which appears in the chiral covariant combinations  $\chi_{\pm} = u^\dagger \chi u^\dagger \pm u \chi^\dagger u$ . In order to recover physical QCD, the external fields are evaluated at the end of the calculation as  $\chi = 2B_0\mathcal{M}$ , being  $\mathcal{M} = \text{diag}(m_u, m_d, m_s)$  the diagonal matrix with the light quark masses.

The pieces of the lagrangian containing the scalar fields are

$$\mathcal{L}_R^{\text{Kin}}(S) + \mathcal{L}_{2S} = \frac{1}{2} \langle \partial^\mu S \partial_\mu S \rangle - \frac{1}{2} M_S^2 \langle S^2 \rangle + 4B_0 c_m \langle S \mathcal{M} \rangle + \mathcal{O}(S^2 \Phi^2, S \Phi^2). \quad (5)$$

The scalar field has therefore a non-zero vacuum expectation value (v.e.v.). In order to define the quantum field theory around the minimum one needs to perform in the scalar field the shift:

$$S = \bar{S} + \frac{4B_0 c_m}{M_S^2} \mathcal{M}, \quad (6)$$

where the shifted scalar nonet fields  $\bar{S}$  has a zero v.e.v. Nonetheless this shift makes  $\bar{S}$  not to be chiral covariant any longer. For the present work this detail is not relevant although other alternative shifts (like  $S = \bar{S} + c_m \chi_+$ ) would restore the explicit covariance. The important detail is that the shift is not equal for all the scalar fields of the multiplet, but proportional to the quark mass matrix  $\mathcal{M}$  and different for each resonance.

The part of the lagrangian containing the vectors, axials and pseudo-scalar resonances,  $\mathcal{L}_R[V, A, P]$ , remains unchanged under the shift but the remaining  $\mathcal{O}(p^2)$  chiral term  $\mathcal{L}_{2\chi}$  and the scalar pieces  $\mathcal{L}_S^{\text{Kin}} + \mathcal{L}_{2S}$  become

$$\mathcal{L}_S^{\text{Kin}} + \mathcal{L}_{2S} + \mathcal{L}_{2\chi} = \mathcal{L}_{\bar{S}}^{\text{Kin}'} + \mathcal{L}_{2\bar{S}}' + \mathcal{L}_{2\chi}' - \frac{8B_0^2 c_m^2}{M_S^2} \langle \mathcal{M}^2 \rangle, \quad (7)$$

yielding a constant term proportional to  $\langle \mathcal{M}^2 \rangle$ , a kinetic term structure for  $\bar{S}$ ,

$$\mathcal{L}_{\bar{S}}^{\text{Kin}'} = \frac{1}{2} \langle \nabla^\mu \left( \bar{S} + \frac{4B_0 c_m}{M_S^2} \mathcal{M} \right) \nabla_\mu \left( \bar{S} + \frac{4B_0 c_m}{M_S^2} \mathcal{M} \right) \rangle - \frac{1}{2} \langle M_S^2 \bar{S}^2 \rangle, \quad (8)$$

an interaction lagrangian without tadpoles,

$$\mathcal{L}_{2\bar{S}}' = c_d \langle \bar{S} u_\mu u^\mu \rangle + c_m \langle \bar{S} (\chi_+ - 4B_0 \mathcal{M}) \rangle \quad (9)$$

and a modified  $\mathcal{O}(p^2)$   $\chi$ PT term,

$$\mathcal{L}'_{2\chi} = \frac{F^2}{4} \langle C_\phi^{-1} [u_\mu u^\mu + \chi'_+] \rangle, \quad (10)$$

provided by the matrix definitions

$$\begin{aligned} \chi'_+ &= \left(1 + \frac{4c_d c_m}{F^2} \frac{4B_0 \mathcal{M}}{M_S^2}\right)^{-1} \left(1 + \frac{4c_m^2}{F^2} \frac{4B_0 \mathcal{M}}{M_S^2}\right) \chi_+, \\ C_\phi^{-1} &= \left(1 + \frac{4c_d c_m}{F^2} \frac{4B_0 \mathcal{M}}{M_S^2}\right). \end{aligned} \quad (11)$$

In order to convert the pNGB kinetic term to the canonical form, one needs to perform a re-scaling  $C_\phi$  on the pNGB fields  $u = \exp(i\Phi/\sqrt{2}F) = \exp(iC_\phi^{\frac{1}{2}}\Phi^{(can)}/\sqrt{2}F)$ .

At LO in  $1/N_C$  the resonance couplings are fixed by the QCD short distance constraints [5, 10]:  $c_d = c_m = F/2$ . Hence, from Eqs. (10) and (11) at LO in  $1/N_C$  one gets for the pion and kaon fields the re-scalings and masses

$$\begin{cases} m_\pi^2 &= 2B_0 \hat{m}, \\ C_\pi &= \left[1 + \frac{2m_\pi^2}{M_S^2}\right]^{-1}, \end{cases} \quad \begin{cases} m_K^2 &= B_0(\hat{m} + m_s), \\ C_K &= \left[1 + \frac{2m_K^2}{M_S^2}\right]^{-1}. \end{cases} \quad (12)$$

### 3.1 $F_\pi$ and $F_K$ at Leading Order

Since the scalar tadpole has been removed, at LO there is only one diagram contributing to the pion decay constant: the tree-level production of the pNGB from the axial current,

$$\langle 0 | \bar{d} \gamma_\mu \gamma_5 u | P^+(p) \rangle = i \sqrt{2} F C_P^{-\frac{1}{2}} p_\mu, \quad (13)$$

and therefore the pion and kaon decay constants are

$$F_\pi = F \left(1 + \frac{2m_\pi^2}{M_S^2}\right)^{\frac{1}{2}}, \quad F_K = F \left(1 + \frac{2m_K^2}{M_S^2}\right)^{\frac{1}{2}}. \quad (14)$$

When  $m_P^2 \ll M_S^2$  the decay constants may be expanded in powers of  $m_P^2$ , recovering the tree-level  $\chi$ PT result  $F_P = F \left[1 + \frac{4L_5}{F^2} m_P^2 + \frac{4L_4}{F^2} (2m_K^2 + m_\pi^2) + \mathcal{O}(m_P^4)\right]$ , with  $L_5 = F^2/4M_S^2$  and  $L_4 = 0$  [5]. This explains why the linear extrapolations work so well. Only near the zero quark mass the LO result in  $1/N_C$  gains sizable non-analytic contributions from the one-loop logarithms (NLO in  $1/N_C$ ):  $F_\pi = F \left[1 + \frac{4L_5^r(\mu)}{F^2} m_\pi^2 - \frac{m_\pi^2}{16\pi^2 F^2} \ln \frac{m_\pi^2}{\mu^2} + \dots\right]$ .

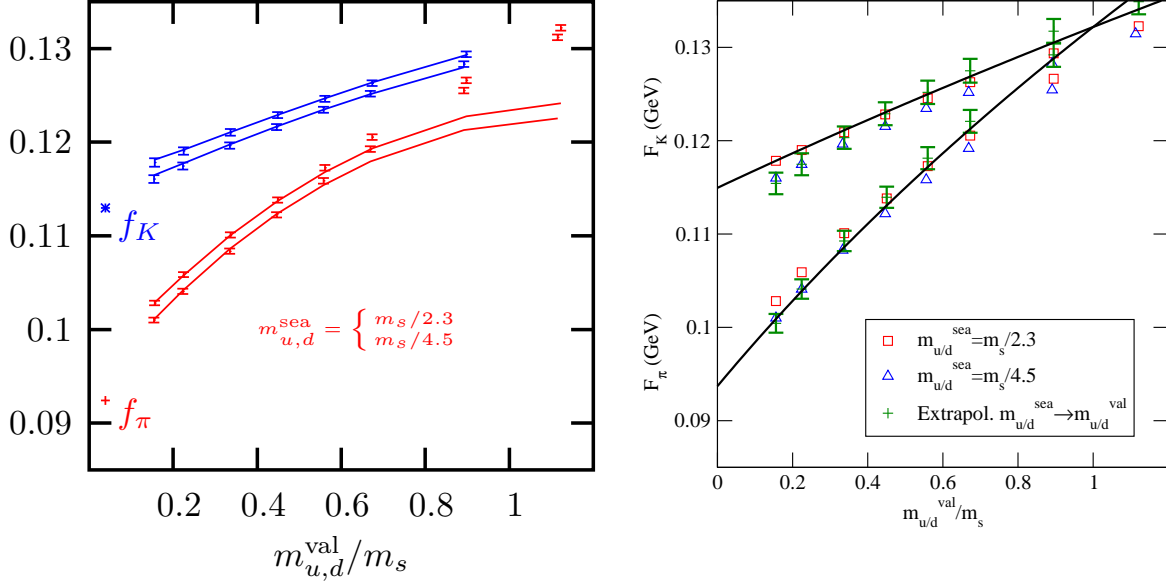


Figure 1: **A)** Lattice results for  $F_\pi$  and  $F_K$  and  $\chi$ PT extrapolations taken from MILC collaboration [7]. **B)** Comparison of the Lattice results [7] with the  $R\chi$ T extrapolations for the values from the fit,  $F = 93.7$  MeV,  $M_S = 1020$  MeV and the splitting  $e_m^S = -0.02$ . The kaon mass varies as  $m_K^2 = (m_K^{Phys})^2 + [m_\pi^2 - (m_\pi^{Phys})^2]/2$  due to its light quark content. They are shown together with the data for  $m_{u,d}^{sea} = m_s/2.3$  (squares),  $m_{u,d}^{sea} = m_s/4.5$  (triangles) and the linear extrapolation to the value  $m_{u,d}^{sea} = m_{u,d}^{val}$  (error bars).

The results for  $F_\pi$  and  $F_K$  from the Lattice simulations (MILC Collaboration [7]) are shown in Fig. (1.A). The simulation handles two kinds of quark masses: The sea-quark masses of the fermions within closed loops; and the valence-quark masses of those which are not from the sea. In this simulation the strange quark valence-mass  $m_s^{val}$  and the strange quark sea-mass  $m_s^{sea}$  take the same value  $m_s = m_s^{val} = m_s^{sea} = 1.14 m_s^{Phys}$ , being  $m_s^{Phys}$  the physical mass of the strange quark. It may be related with the physical kaon mass through Eq. (12). The  $u/d$  quark valence-masses  $m_{u,d}^{val}$  (isospin limit is assumed) are varied continuously between nearly zero and the physical mass of the strange quark. Finally, the simulation is run for two values of the sea-masses of the quarks  $u/d$ :  $m_{u,d}^{sea} = m_s/2.3$  and  $m_{u,d}^{sea} = m_s/4.5$  (squares and triangles respectively in Fig. (1.B)).

The modifications due to the sea quark mass are much smaller than those from the valence masses, as it is expected in the large  $N_C$  limit, since the closed quark loops would be suppressed by  $1/N_C$ . Thus, in this work I have generated the matrix elements for equal values of the sea and valence masses, i.e. for  $m_{u,d}^{sea} = m_{u,d}^{val}$ , through a simple linear extrapolation from the samples  $m_{u,d}^{sea} = m_s/2.3$  and  $m_{u,d}^{sea} = m_s/4.5$  (error bars in Fig. (1.B)).

The theoretical expressions derived from  $R\chi$ T for  $F_\pi$  and  $F_K$  in Eqs. (14) were fitted

to these extrapolated points. The error in my input data in the fit was a 1 % error, a typical discretization error, but it did not account the uncertainties in the  $m_{u,d}^{sea}$  extrapolation to the value  $m_{u,d}^{sea} = m_{u,d}^{val}$ . The variation of the decay constant when changing  $m_{u,d}^{sea}$  could be considered as a rough estimate of this uncertainty. In addition, the errors due to NLO contributions in  $1/N_C$  have not been considered. The fit yields the values  $F = 94.1 \pm 0.9$  MeV and  $M_S = 1049 \pm 25$  MeV, with  $\chi^2/\text{dof} = 11.0/13$ . This yields for the physical pion and kaon decay constants  $F_\pi = 95.8 \pm 0.9$  MeV and  $F_K = 113 \pm 1.4$  MeV, in acceptable agreement with the experimental values  $F_{\pi^+} = 92.4 \pm 0.07 \pm 0.3$  MeV and  $F_{K^+} = 113.0 \pm 1.0 \pm 0.3$  MeV [11]. One may also estimate the  $\chi$ PT coupling at LO in  $1/N_C$ ,  $L_5 = F^2/4M_S^2 = (2.01 \pm 0.10) \cdot 10^{-3}$ .

Nevertheless, here the chiral logarithms are lacking. In  $\chi$ PT they produce an important non-analytic effect and a large bending in the  $F_\pi$  curve, since its slope becomes large at small pion mass due to  $\frac{dF_\pi}{dm_\pi^2} \sim \ln \frac{m_\pi^2}{\mu}$ . The effect of the logs on  $F_K$  is much more reduced since the value of the kaon mass does not become small when  $m_{u,d} \rightarrow 0$ . That is the reason for the better agreement of the  $F_K$  result. Eventually the NLO calculation in  $1/N_C$  (one loop) would introduce this extra non-analytic curvature and the usual one loop result for  $F_\pi$  in  $\chi$ PT would be recovered.

The mass splitting between the masses of the two  $I = 0$  scalar resonances can also be studied since they do not contribute equally to  $F_\pi$  and  $F_K$ . The discussion is developed in the appendix. At LO in  $1/N_C$  there would be two mass eigenstates with square masses  $\bar{M}_{S_n}^2 \equiv M_S^2 - 8e_m^S B_0 m_{u,d}$  and  $\bar{M}_{S_s}^2 \equiv M_S^2 - 8e_m^S B_0 m_s$ , with quark contents  $(\frac{1}{\sqrt{2}}\bar{u}u + \frac{1}{\sqrt{2}}\bar{d}d)$  and  $\bar{s}s$  respectively. The pion and kaon decay constants depend now on the splitting parameter  $e_m^S$ . However the fit to the former data is not sensitive to this coupling, yielding:  $F = 93.7 \pm 1.5$  MeV,  $M_S = 1020 \pm 80$  MeV and  $e_m^S = -0.02 \pm 0.05$ , with similar  $\chi^2/\text{dof} = 10.9/12$ . It provides an estimate of the scalar masses at large  $N_C$ : one gets the values  $\bar{M}_{S_n} = 1020 \pm 80$  MeV and  $\bar{M}_{S_s} = 1040 \pm 90$  MeV, highly correlated, and their splitting, equal to  $\bar{M}_{S_s} - \bar{M}_{S_n} = 20 \pm 40$  MeV. In addition, large OZI-rule violations could occur at NLO of the same size as the quark mass corrections. In the worst case, one would expect them to be of the order of  $1/N_C \simeq 33\%$  times the value of the scalar mass at LO in  $1/N_C$ .

The R $\chi$ T extrapolation is shown in Fig. (1.B) (solid line). Since this effective field theory is for equal valence and sea masses, the theoretical result is compared with an emulation of the Lattice data for  $m_{u,d}^{sea} = m_{u,d}^{val}$ , obtained by linear extrapolation as it was explained before.

## 4 Conclusions

R $\chi$ T has been shown to be an interesting tool to analyse the Lattice data, which usually are generated for non-physical values of the light quark masses. The present study hints that the light mesonic resonances may play an important role in the large quark mass extrapolations. This work explores the pion and kaon decay constants, providing successful results. Its

importance and aim is not just the determination of the decay constants but to present an alternative idea to interpret the Lattice simulations for large unphysical values of the masses, giving a clear explanation of why the usual linear extrapolations yield such a good result. Likewise, this provides solid theoretical foundations based on the underlying QCD for these techniques. Thus, the  $1/N_C$  expansion might be as well a suitable framework to describe the heavy quark matrix elements ( $f_B$ ,  $B_B$ ...) at large values of the  $u/d$  quark masses, where a similar linear behaviour has been also observed.

The fact that at low energies  $R\chi T$  recovers  $\chi PT$  ensures that we are introducing the proper low mass behaviour [3, 4, 6]. Former works [8, 9] noticed the necessity of a separation scale  $\Lambda_{cut}$  where the  $\chi PT$  loops become irrelevant. The resonance masses provide a “natural” scale where the chiral extrapolations fails and where the dynamics of the observable changes drastically.

The fits to the simulations were done for an emulation of the Lattice data, obtained by extrapolating  $m_{u,d}^{sea}$  to the value  $m_{u,d}^{sea} = m_{u,d}^{val}$ , that varied in a wide and continuous range between zero and the strange quark mass. For a more proper analysis one would need a simulation with equal sea and valence masses. However, the main dependence comes from the valence-quarks and the sea-quark effects are small, since the closed quark loops are suppressed by  $1/N_C$ . Therefore, the present calculation can be considered an adequate estimate of the hadronic parameters. From the former fit the values  $F = 94.1 \pm 0.9$  MeV and  $M_S = 1049 \pm 25$  MeV were obtained, together with the  $\chi PT$  coupling estimate at LO in  $1/N_C$ ,  $L_5 = (2.01 \pm 0.10) \cdot 10^{-3}$ . The scalar mass splitting showed large uncertainties.

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## Appendix: Scalar mass splitting

The resonance multiplet has been considered at first as degenerate in mass. Nonetheless the resonance masses can gain contributions due to the quark masses. In the large  $N_C$  limit, Chiral invariance requires that at order  $\mathcal{O}(m_q)$  the mass splitting comes only through a chiral invariant term [12]:

$$\Delta\mathcal{L}_{m_q} = e_m^S \langle \chi_+ S^2 \rangle, \quad (15)$$

being  $e_m^S$  an  $\mathcal{O}(N_C^0)$  dimensionless constant, independent of the quark masses.

The shift now in the scalar field to re-absorb the tadpole is slightly different:

$$S = \bar{S} + 4B_0 c_m \mathcal{M} \left[ M^2 - 8e_m^S B_0 \mathcal{M} \right]^{-1}. \quad (16)$$

The mass eigenvalues for the  $I = 0$  scalars are not  $M_S^2$  anymore but the two values  $\bar{M}_{S_n} \equiv M_S^2 - 8e_m^S B_0 m_{u,d}$ , for the state  $\left(\frac{1}{\sqrt{2}}\bar{u}u + \frac{1}{\sqrt{2}}\bar{d}d\right)$ , and  $\bar{M}_{S_s} \equiv M_S^2 - 8e_m^S B_0 m_s$ , for  $\bar{s}s$ . Nonetheless, the physical scalar states will separate away from this ideal mixing and these masses will gain also contributions due to NLO effects in  $1/N_C$ .

At LO in  $1/N_C$  the pNGB masses still remain as given in Eq. (12). However the re-scaling factors for pions and kaons change accordingly:

$$C_\pi = \left[ 1 + \frac{2m_\pi^2}{M_S^2 - 4e_m^S m_\pi^2} \right]^{-1}, \quad C_K = \left[ 1 + \frac{2m_K^2}{M_S^2 - 4e_m^S m_K^2} \right]^{-1}, \quad (17)$$

which therefore modify the pion and kaon decay constants.

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